Monetary Policy as an Optimal Control Problem[#]

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1. Introduction

The conduct of monetary policy can be considered as an optimal control task. As such, it has been an intensive research subject for many years. In the framework of macroeconometric models, the study of optimal monetary policy can be found, for example, in Chow's work (1976). Today, under the dominance of the New Keynesian models, which is seen as the theoretical background for monetary policy, this study has re-emerged, see Levin and Williams (2003), Svensson and Tetlow (2006) and Orphanides and Williams (2008). As the position of New Keynesian models is so prevalent, every self-respected central bank in developed countries often claims that these models are their fundamental analytical tool of their monetary policy, and the final monetary policy decisions are based on the outputs of these models. In this regard, the Czech National Bank is no exception (see Anderle et al., 2009).

In the literature, New Keynesian models often are presented in a discontinuous time fashion (Galí, 2008) which may have some advantages, but in a discrete time framework it has one disadvantage: discrete dynamic theory is not convenient for qualitative analysis of solutions (see Glass et al, 2008) time. To fill this gap in the literature, we propose a continuous time version of the New Keynesian model and investigate the impact of monetary policy conducted according to the loss function or to the Taylor rule by a central bank. In order to be able to do so, first we derive a deterministic and continuous non-linear two-equation New-Keynesian model. One of them is the IS equation of commodity market with a logistic investment function to make the dynamics richer.

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The second one is the Phillips curve which connects both the real and nominal sector of the economy. Monetary policy in this model is performed in the inflation targeting regime first according to the loss function and then according to the Taylor rule.

To show how such a model would work, we give two examples of their functioning. First we construct the optimal control problem using minimization of loss function. The problem formulated by this approach is a non-linear one. As such it can be solved by using the Pontryagin's principle, whose theoretical background will be briefly discussed later. Since solving a non-linear control problem with the Pontryagin's principle can be very difficult, sometimes even unsolvable, for this purpose we suggest the use of fuzzy control. Fuzzy control is a new control approach which may succeed when other traditional control methods are unable to deal with. An overview of possible applications in technical and other areas can be found in Driankov et al (1996) and Novak (2000). The application of fuzzy control in economics can be found in the work of Kukal and Tran Van Quang (2013).

Then as an alternative way to investigate monetary policy as a control problem, we establish the model with the same structure of dynamics with the use of a modified Taylor rule. Taylor rule is a reaction function of a central bank to an actual state of an economy. Unlike the case with a loss function, which is a typical optimization problem, monetary policy in inflation targeting regime with the use of the Taylor rule is conducted in such a way that interest rate is continuously manipulated in order to get the whole system to reach a desirable state. This is in a sharp contrast to the case with the loss function. In this case, our control problem corresponds to the bang-bang principle. Here the interest rate is pre-determined to move inside an interval. The optimal solution of the problem then requires the interest rate to switch from its maximum to minimum and vice versa. We will show how these two cases work through the numerical examples.

2. Reduced form of New-Keynesian Model

The dynamic of an economy with an active central bank can be described by two differential equations. The first one represents the dynamics of commodity market. The original mathematical description of commodity market is the continuous *IS-LM* model (Kodera and Málek, 2008) as follows:

$$\dot{Y} = \omega\big[I\big(Y, R(t) - \pi(t)\big) - S(Y(t), R(t) - \pi(t))\big],$$

where Y = production,
I = investment,
R = nominal interest rate,
S = saving quantity,

I and S are functions of Y, R and π. The dynamics of production is generated in this way. Let's suppose $S>I$. It means the right hand side of the equation is negative. Then the left hand side must go negative too and *vice versa.* Dividing the above equation by Y, we get:

$$\frac{\dot{Y}(t)}{Y(t)} = \omega\left[\frac{I(Y(t), R(t) - \pi(t))}{Y(t)} - \frac{S(Y(t), R(t) - \pi(t))}{Y(t)}\right].$$

As $y(t) = \log Y(t)$, we can rearrange the equation above as:

$$\dot{y}(t) = \omega\left[\frac{I(e^{y(t)}, R(t) - \pi(t))}{e^{y(t)}} - \frac{S(e^{y(t)}, R(t) - \pi(t))}{e^{y(t)}}\right].$$

Denoting $\frac{I(.)}{e^y} = i(.), \frac{S(.)}{e^y} = s(.)$, the dynamics of production can be expressed as follows:

$$\dot{y}(t) = \omega\big[i\big(y(t), R(t) - \pi(t)\big) - s\big(y(t), R(t) - \pi(t)\big)\big], \qquad (1)$$

where $\alpha > 0$,

and i and s are the so called propensity to invest function and propensity to save function respectively, but from now on we will call them as an investment and saving function for short.

The second behavioral equation in our model is the Phillips curve in a slightly different form:

$$e^{\pi^f(t)} = \left[\frac{V(t)}{V_0}\right]^{\gamma}, \qquad (2)$$

where $\pi^f(t)$, stand for the fundamental[1] rate of inflation and rate of employment respectively, V_0 is the rate of employment under zero

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[1] More precisely, the rate of inflation is defined by equation (2).

inflation. The firms' demand of for labour is determined from the inversion of one factor production function

$$Y(t) = F(L(t)),$$

where $Y(t)$, $L(t)$ denotes production and level of employment respectively. The rate of employment is defined as labour demand-labour supply ratio, so we obtain:

$$V(t) = \frac{L(t)}{N} = \frac{F^{-1}(Y(t))}{N}, \tag{3}$$

where F^{-1} denotes the inversion of production function and N is the households' labour supply. In this model, labour supply is constant. Let us assume a geometrical adjustment process of actual inflation to its fundamental rate as:

$$\dot{\pi}(t) = \omega[\pi^f(t) - \pi(t)].$$

Using (3) in equation (2) and substituting in the above equation, after taking logarithms and some rearrangement, we get:

$$\dot{\pi}(t) = \omega\{\gamma[g^{-1}(y(t)) - n - v_0] - \pi(t)\}, \tag{4}$$

where $g^{-1}(y(t)) = \log F^{-1}(e^{y(t)})$, $y(t) = \log Y(t)$, $n = \log N$, and $v_0 = \log V_0$. Equations (1) and (4) constitute an economic dynamic system which generates trajectories of production and inflation.

3. Logistic Investment Function

To make the dynamics of variables in the model more complex, we modify the investment function in the following way. Let $u(R(t) - \pi(t))$ be reciprocal function of real interest rate:

$$u(R(t) - \pi(t)) = \frac{h}{R(t) - \pi(t)},$$

and production y is a logistic function

$$l(y(t)) = \frac{1}{1 + ce^{-by(t)}}.$$

Investment function then is a product of u and y as

$$i(y(t), R(t) - \pi(t)) = u(R(t) - \pi(t))l(y(t)$$

$$= \frac{h}{(R(t) - \pi(t))(1 + ce^{-by(t)})}. \tag{5}$$

The savings function is assumed to be linear in production and real interest rate

$$s(y(t), R(t) - \pi(t)) = s_0 + s_1 y(t) + s_2(R(t) - \pi(t)). \tag{6}$$

where h, b, c in (5) s_0, s_1 and s_2 in (6) are positive parameters. Plugging (5) and (6) into equation (1), we have

$$\dot{y}(t) = \omega[J(t) - s_0 - s_1 y(t) - s_2(R(t) - \pi(t))], \omega > 0. \tag{7}$$

where $\quad J(t) \quad = \frac{h}{(R(t) - \pi(t))(1 + ce^{-by(t)})}.$

We assume that in equation (4) production function is one-factor Cobb Douglas function as follows:

$$Y(t) = AL^{1-\alpha}(t).$$

Then the demand for labour is the inversion of Cobb-Douglas production function

$$L(t) = [A^{-1}Y(t)]^{\frac{1}{1-\alpha}}.$$

Taking logarithm of both sides, we obtain

$$g^{-1}(y(t)) = l(t) = \frac{1}{1-\alpha}(y(t) - a).$$

Substituting it into equation (4), we have

$$\dot{\pi}(t) = \beta \left\{ \gamma \left[\frac{1}{1-\alpha}(y(t) - a) - n - v_0 \right] - \pi(t) \right\}, \beta > 0, \tag{8}$$

The system of equations (7) and (8) is a continuous time version of New-Keynesian model.

4. Loss function

Inflation targeting problem is often specified as an optimal control problem. In this case behavior of the central bank is usually described by New Keynesian economics loss function. Mainly it is based on the shape of function introduced authors Barro and Gordon (1983):

$$U(y(t), \pi(t)) = \left(y(t) - y_g\right)^2 + \left(\pi(t) - \pi_g\right)^2.$$

As we decided for continuous approach to this problem, time variable t obtains values from infinite interval $[0, \infty)$ and optimization in infinite horizon is expressed by minimization of improper integral of loss function

$$J(y, \pi) = \int_0^\infty U(y(t), \pi(t)) e^{-\rho t} dt =$$

$$\int_0^\infty \left[(\pi(t) - \pi_g)^2 + (y(t) - y_g)^2\right] e^{-\rho t} dt.$$

(9)

Relations (7), (8) and (9) constitute continuous optimal control problem in infinite horizon.

5. Zero Inflation Steady State

The steady state solution of the problem (1), (4) and (9) is the solution, where state and control variables do not change in time, let $y(t) = \bar{y}, \pi(t) = 0, \; R(t) = \bar{R}$. It is essential for the structure of the task. Keeping these variables unchanged, the mentioned problem becomes a problem:

Minimize

$$J(y, \pi) = \int_0^\infty \left[(\bar{\pi} - \pi_g)^2 + (\bar{y} - y_g)^2\right] e^{-\rho t}$$

$$= \frac{1}{\rho} \left[(\bar{\pi} - \pi_g)^2 + (\bar{y} - y_g)^2\right].$$

Subject to

$$0 = i(\bar{y}, \bar{R} - \bar{\pi}) - s(\bar{y}, \bar{R} - \bar{\pi}). \tag{10}$$

$$0 = \frac{1}{1-\alpha}(\bar{y} - a) - n - v_0 - \bar{\pi}. \tag{11}$$

We construct known Lagrangian for steady state problem

$$L = \frac{1}{\rho}\left[(\bar{\pi} - \pi_g)^2 + (\bar{y} - y_g)^2\right]$$
$$+ \mu_1 \alpha[i(\bar{y}, \bar{R} - \bar{\pi}) - s(\bar{y}, \bar{R} - \bar{\pi})]$$
$$+ \mu_2 \left[\beta\left(\frac{1}{1-\alpha}(\bar{y} - a) - n - v_0\right) - \bar{\pi}\right]$$

Differentiating with respect to $\bar{y}, \bar{\pi}, \bar{R}$, denoting $\lambda_1 = \rho\mu_1$, $\lambda_2 = \rho\mu_2$, we get

$$L_y = 2(\bar{y} - y_g) + \lambda_1\left[\frac{\partial i(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{y}} - \frac{\partial s(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{y}}\right] + \lambda_2 \frac{1}{(1-\alpha)} = 0, \qquad (12)$$

$$L_\pi = 2(\bar{\pi} - \pi_g) + \lambda_1\left[\frac{\partial i(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{\pi}} - \frac{\partial s(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{\pi}}\right] - \lambda_2 = 0, \qquad (13)$$

$$L_R = \lambda_1\left[\frac{\partial i(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{R}} - \frac{\partial s(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{R}}\right] = 0. \qquad (14)$$

As $\frac{\partial i(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{R}} < 0$ and $\frac{\partial s(\bar{y}, \bar{R} - \bar{\pi})}{\partial \bar{R}} > 0$, $\lambda_1 = 0$, what results from equation (14). Further, from (12) and (13) we get

$$\bar{y} = y_g - \frac{1}{2}\lambda_2\frac{1}{1-\alpha}, \quad \bar{\pi} = \pi_g + \frac{1}{2}\lambda_2. \qquad (15)$$

Unknown quantities $\bar{R}, \lambda_2$ are computed from equations (10), (11). From equation (11) we express λ_2:

$$\lambda_2 = 2\frac{\frac{1}{1+\alpha}(y_g - a) - n - v_0}{\left(\frac{1}{1+\alpha}\right)^2 + 1}. \qquad (16)$$

From (15) we get

$$\bar{y} = y_g - \frac{\frac{1}{1+\alpha}(y_g - a) - n - v_0}{\frac{1}{1+\alpha} + 1 + \alpha}, \qquad (17)$$

$$\bar{\pi} = \pi_g + \frac{\dfrac{1}{1+\alpha}(y_g - a) - n - v_0}{\left(\dfrac{1}{1+\alpha}\right)^2 + 1}.$$

Steady state problem has simple optimal solution $\bar{y}, \bar{\pi}$ given by (17) and $\bar{R}$ given by (5). Solving this problem and considering zero inflation steady state as an inflation target, we get instead of (17)

$$\bar{y} = y_g, \quad \bar{\pi} = \pi_g$$

Steady state solution of the model is very important because it is usually the final continuing strategy in a model of optimal control with infinite horizon.

6. Pontryagin principle

To solve and analyse we choose Pontryagin principle as a relatively simple method for reaching of our objectives. For the solution of the problem given by equations (1)-(3) Pontryagin principle (Pontryagin et al. (1976), Jahn (2007)) is used. The Hamiltonian of the problem has a form

$$H(y, \pi, R) = -e^{-\rho t}\left[\left(y(t) - y_g\right)^2 + \left(\pi(t) - \pi_g\right)^2\right] +$$

$$+\psi_1(t)\alpha\left[i\left(y(t), R(t) - \pi(t)\right) - s\left(y(t), R(t) - \pi(t)\right)\right] \quad (18)$$

$$+\psi_2(t)\beta\left\{\gamma\left[\frac{1}{1-\alpha}(y(t) - a) - n(t) - v_0\right] - \pi(t)\right\}.$$

The Pontryagin principle among others states that if the solution of an optimal control problem minimizes objective functional, then the Hamiltonian reach maximum as a function of R. But the expression $\left[i\left(y(t), R(t) - \pi(t)\right) - s\left(y(t), R(t) - \pi(t)\right)\right]$ decreases in R, so the Hamiltonian reaches its maximum in R_0 or in R_1 which depends on positivity or negativity of $\psi_1(t)$. The optimal control solution thus takes either the extreme values of control variable and is called the bang-bang optimal control solution.

The co-state equations are

$$\dot{\psi}_1(t) = 2[y(t) - y_g]e^{-\rho t} - \beta\gamma\frac{1}{1-\alpha}\psi_1(t),$$

$$-\psi_2(t)\left[\frac{\partial i}{\partial y}\big(y(t),R(t)-\pi(t)\big)-\frac{\partial s}{\partial y}\big(y(t),R(t)-\pi(t)\big)\right],$$

$$\dot{\psi}_2(t)=2[\pi(t)-\pi_g]e^{-\rho t}-\beta\gamma\psi_1(t).$$

$$-\psi_2(t)\left[\frac{\partial i}{\partial \pi}\big(y(t),R(t)-\pi(t)\big)-\frac{\partial s}{\partial \pi}\big(y(t),R(t)-\pi(t)\big)\right].$$

As the system of co-state equations is a non-autonomous one, to re-arrange it by multiplying the whole system with $e^{\rho t}$, we get

$$e^{\rho t}\dot{\psi}_1(t)=-2\big[y(t)-y_g\big]-\beta\gamma e^{\rho t}\psi_2(t)$$

$$-e^{\rho t}\psi_1(t)\left[\frac{\partial i}{\partial y}\big(y(t),R(t)-\pi(t)\big)-\frac{\partial s}{\partial y}\big(y(t),R(t)-\pi(t)\big)\right]$$

$$e^{\rho t}\dot{\psi}_2(t)=-2\big[\pi(t)-\pi_g\big]-\beta\gamma e^{\rho t}\psi_2(t)$$

$$-e^{\rho t}\psi_1(t)\left[\frac{\partial i}{\partial \pi}\big(y(t),R(t)-\pi(t)\big)\right.$$

$$\left.-\frac{\partial s}{\partial \pi}\big(y(t),R(t)-\pi(t)\big)\right]$$

Let us define $p_j(t)=e^{\rho t}\psi_j(t)$, j=1, 2. Taking its derivative with respect to t we get

$$\dot{p}_j(t)=\rho e^{\rho t}\psi_j(t)+e^{\rho t}\dot{\psi}_j(t).$$

Plugging it to (18) and the co-state system equations, we get the Hamiltonian multiplied by $e^{\rho t}$ and new transformed co-state equations.

$$e^{\rho t}H(y,\pi,R)=H_\rho(y,\pi,R)$$
$$=\left[\big(y(t)-y_g\big)^2+\big(\pi(t)-\pi_g\big)^2\right]+$$
$$+p_1(t)\alpha\big[i\big(y(t),R(t)-\pi(t)\big)-s\big(y(t),R(t)-\pi(t)\big)\big]$$
$$+p_2(t)\big[\gamma\pi(t)+\beta\big(y(t)-y_g\big)\big],$$

(19)

$$\dot{p}_1(t) = 2y(t) - \gamma \frac{1}{1-\alpha} p_2(t).$$

$$
\begin{aligned}
- p_1(t) \Big[&\frac{\partial i}{\partial y}(y(t), R(t) - \pi(t)) \\
&- \frac{\partial s}{\partial y}(y(t), R(t) - \pi(t)) - \rho \Big],
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\dot{p}_2(t) = \pi(t) &- (\beta\gamma - \rho)p_2(t) \\
&- p_1(t) \Big[\frac{\partial i}{\partial \pi}(y(t), R(t) - \pi(t)) \\
&- \frac{\partial s}{\partial \pi}(y(t), R(t) - \pi(t)) \Big].
\end{aligned}
\tag{21}
$$

The system of equations (19), (20) and (21) will be used to find the optimal solutions in the next section.

Exclusion of evidently non-optimal regimes

Solving analytically the whole system described by equations (19) – (21) is very complicated. Therefore, we only use the analytical approach to exclude some trajectories which are not optimal. Let begin with assumption $y > y_g$. First we prove that it cannot be optimal to keep the interest rate at level R_0 under condition $y > y_g$. Let us assume the opposite statement, i.e. keeping the interest rate at R_0. In accordance with Pontryagin principle R_0 should maximize H_ρ, which imply $p_1(t) > 0$. Let

$$A(t) = -\frac{\partial i}{\partial y}(y(t), R(t) - \pi(t)) + \frac{\partial s}{\partial y}(y(t), R(t) - \pi(t)) + \rho,$$

$$c = -(\gamma - \rho),$$

$$B(t) = \frac{\partial i}{\partial \pi}(y(t), R(t) - \pi(t)) - \frac{\partial s}{\partial \pi}(y(t), R(t) - \pi(t)).$$

Using the above expressions the system (20), (21) takes a form:

$$\dot{p}_1(t) = 2(y(t) - y_g) + p_1(t)A(t) - \beta p_2(t),$$

$$\dot{p}_2(t) = 2\pi(t) - p_1(t)B(t) - (\gamma - \rho)p_2(t).$$

Linearizing it in the point $(y, \pi, R, p_1, p_2) = (y_g, 0, R_0, 0, 0)$ we get:

$$\dot{p}_1(t) = p_1(t)A - \beta p_2(t), \tag{22}$$

$$\dot{p}_2(t) = -p_1(t)B + cp_2(t), \tag{23}$$

where
$$A = -\frac{\partial i}{\partial y}(y_g, R_0) + \frac{\partial s}{\partial y}(y_g, R_0) + \rho,$$
$$c = c = -(\gamma - \rho),$$
$$B = \frac{\partial i}{\partial \pi}(y_g, R_0) - \frac{\partial s}{\partial \pi}(y_g, R_0).$$

The characteristic equation of the linearized system (22), (23) is

$$\lambda^2 - (A + c)\lambda + Ac - \beta B = 0.$$

The roots of characteristic equations are positive provided that $Ac - \beta B > 0$, thus the state in time t_1 is not accessible, because $p_1(t) > 0$, and $\dot{p}_1(t) > 0$ for $t < t_1$. Let $Ac - \beta B < 0$, $p_1(t) > 0$. Assume that $p_1(t)$ decrease to 0. In this case $p_2(t)$ increase and according to equation (22) $p_2(t) > 0$ and thus cannot reach 0. We can analogously prove that keeping R at R_1 cannot be optimal. As obtaining a complete analytical solution is theoretically challenging and time consuming we suppose that using fuzzy regulator to get an optimal solution is more appropriate approach in this case.

7. Using Fuzzy Regulator

The principle of fuzzy regulator is not so complex. Working with a fuzzy regulator system only requires to understand the basic principles of how to control this system. These basic principles are called *control strategy* (for more detailed explanation of this concept, see Driankov (1993) and Novák (2000)). Fuzzy regulator is a set of predicates of type If - Then. These predicates are formulated in natural language. Let us illustrate the use of fuzzy regulator in a simple model of central bank. Suppose that a central bank implements its chosen monetary policy with the inflation targeting regime. The central bank would change the short-term interest rate one way or the other way whenever it recognizes that there would be some substantial future deviation of the product and inflation from its targets and the change in the short-term interest rate does not affect its primary task: keeping the price stability in the economy. Optimal control of analyzed system requires using bang-bang regulator. It is considerable advantage lying in the fact that if the central bank changes the interest rate it does not need to calculate the size of the change. For the central bank the bounds of the interest rate should be

known and the bank can set the interest rate either close to the lower bound or the upper bound. The list of optimal moves of interest rate taken by a central bank in the bang-bang fuzzy regulation framework is shown in Tab 1.

Tab. 1: The list of optimal moves of interest rate

Difference y-y_n	Difference π-π_g	Control variable R
Big and negative	Big and negative	Lower bound R
Small and negative	Small and negative	No small
Zero	Zero	Use previous experience
Small and positive	Small and positive	No big
Big and positive	Big and positive	Upper bound R

Now let's proceed with a numerical model mimicking a real economy. We assume that the central bank targets potential product y_n and zero inflation. Thus:

$$i(y, R - \pi) = \frac{a}{\left(1 + R - \pi(t)\right)\left(1 + e^{-by(t)}\right)},$$

$$s(y, R - \pi) = s_0 + s_1 y(t) + s_2 \left(R - \pi(t)\right).$$

For this economy we suppose that in accordance with reality, this set of parameter values is valid: $\alpha = 0.87$; $a = 0.42$; $b = 6$; $s_0 = 0.12$; $s_1 = 0.8$; $s_2 = 1.6$; $\lambda = 0.05$; $y_n = 0$; $\rho = 0.05$; $R_0 = 0.04$; $R_1 = 0.06$;

Plugging them into equations (1)-(4) we get:

$$J(y, \pi) = \int_0^\infty \left[\pi(t)^2 + y(t)^2\right] e^{-0.05t},$$

$$\dot{y}(t) = \alpha \left[\frac{0.42}{(1+R-\pi(t))(1+e^{-6y(t)})} - 0.12 - 0.8y(t) - 1.6\left(R - \pi(t)\right)\right] \dot{\pi}(t) = \beta \left\{\gamma \left[\frac{1}{1-\alpha}(y(t) - a) - n - v_0\right] - \pi(t)\right\}, \ \beta > 0,$$

$$0.045 \leq R \leq 0.06.$$

The initial conditions are $y(0) = 0.5, \pi(0) = 0.06$.

Suppose that central bank evaluates its policy at the end of each quarter and implements the approved policy after two months. It switches interest rate after two month whenever find that the difference $y - y_n$ have changed its sign. When solving the optimal control problem with infinite horizon by the bang-bang principle, we use the boundary interest rates for pre-stationary control until the steady state is reached. The same strategy we choose when using fuzzy regulator. We start at time $t = 0$ with $y = 0.5$ which is above its potential, so we set the interest rate at maximum 0.06. As the result, the production decreases. After 18 periods it reaches its potential y_0. As we assume it takes other 2 periods for us to make the decision about the interest rate. At period 20, as the production is under its potential, we set the interest rate at its minimum of 0.045. With a low

Figure 1: Smoothing output gap by using a fuzzy regulator
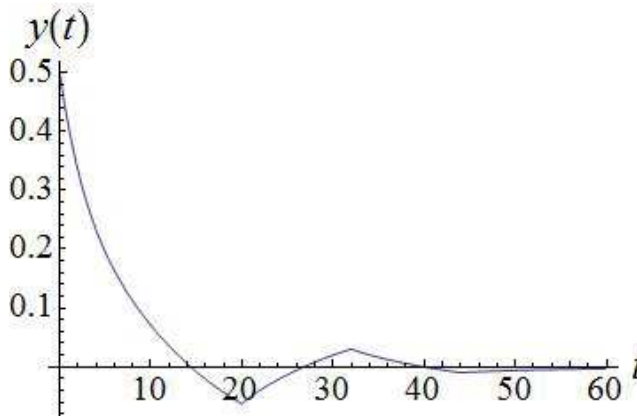
Interest starts increasing and overpasses its potential at period 27. According to the assumption we make at the beginning, it takes another quarter for the central bank to evaluate its rate, the production monetary policy and 2 periods to reset the interest rate. Since the production is very close to the potential product, there is no need to set the interest rate at maximum and minimum. So, at period 32, we set the interest rate at 0.052 at the middle of the interval. The change of interest rate causes the reversion of the product to its potential. As the product moves closely around its potential, the central bank only needs to fine-tune the interest rate to steer the product so that it converges smoothly to it potential. In our case the interest rate needs to be set at the value of 0.05. Fig. 1 shows how the product converges to its potential using fuzzy regulator. The vertical axis of the figure shows the size of the output gap and the horizontal axis represents the time in months. As far as the difference between the real inflation and the inflation target is concerned, by the

nature of the Phillips curve, the inflation adapts very quickly and after several periods it approaches zero and stays unchanged from then on. Therefore, this simple model shows that using the Pontryagin principle combined with a fuzzy regulator, a central bank using an inflation targeting regime can reach its objectives in a relatively comfortable way.

8. A modified Taylor rule

The other approach to solve the problem of inflation targeting is with help of reaction function which is obviously given by Taylor rule. In our model, we have a central bank which tries to actively stabilize the economy to reach zero inflation in steady state. Its monetary policy is to target the real inflation rate by following a modified Taylor rule (1993) in this form:

$$R(t) = \bar{R} + \theta_y y(t) + \theta_\pi \pi(t), \; \theta_y > 0, \qquad \theta_\pi > 0, \tag{24}$$

where $\bar{R}$ is the interest rate in steady state, θ_y and θ_π are parameters. As the system of differential equations (1) and (4) has 3 unknowns $R(t)$, $y(t)$ and $\pi(t)$, first we have to exogenously determine R so that the zero inflation steady state can be reached. Then the solutions of steady state denoted as $(\bar{y}, \bar{R})$, meaning that in this state the production is time invariant and inflation is a zero, must be the solutions of the system of two equations (1) and (4):

$$0 = \alpha[i(y, R) - s(y, R)], \qquad \alpha > 0$$
$$= \omega\{\gamma[g^{-1}(y) - n - v_0].$$

9. Numerical example

To show how the model works, we need to find a set of appropriate values for parameters included in the model. The main problem we face is that not all of them are observable, and others may vary with time. Many of them have not been tested in the literature so far, and it makes their choice some time very speculative.

In equation (1) of investment function, we chose following numerical values $h = 0.42$, $b = 10$, $c = 1$. The parameters of savings function are $s_0 = 0.12$; $s_1 = 0.8$, $s_2 = 1.6$, $R = 0.05$. The adjustment parameter $\omega = 4.2$. In equation (4) we calibrate its parameters as follows. In production function they are $\alpha = 0.4$, $a = 0$. For the labour supply, we

have $n = 0$. The natural rate of employment $v_0 = 0$ and adjustment parameter of Phillips curve $\gamma = 0.1$. Adjustment parameter of differential equation (4) $\beta = 0.1$. Putting them in equations (1) and (4), we obtain

$$\dot{y}(t) = \omega\big[J - 0.12 - 0.8y(t) - 1.6\big(0.05 - \pi(t)\big)\big] \qquad (25)$$
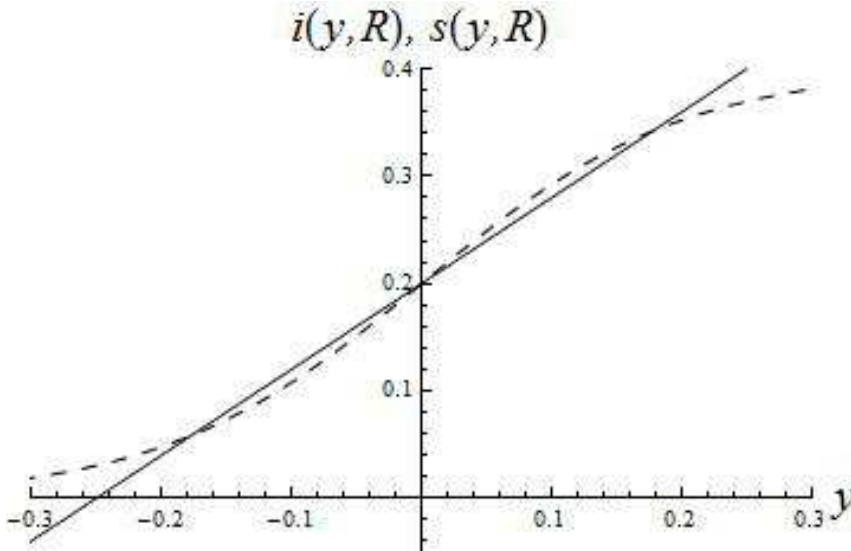
$$\dot{\pi}(t) = 0.1\Big\{0.1\Big[\frac{1}{1 - 0.4}y(t)\Big] - \pi(t)\Big\} \qquad (26)$$

where $J = \dfrac{0.42}{\big(0.05 - \pi(t)\big)\big(1 + e^{-10y(t)}\big)}$.

As we choose $\pi(t) = 0$, equations (10) and (11) become:

$$i(y, 0.05) = \frac{0.42}{(0.05 - \pi(t))(1 + e^{-10y})}$$
$$- 0.12 - 0.8y(t) - 1.6\big(0.05 - \pi(t)\big).$$

Figure 2 Investment and saving functions

In figure 2, the graphs of investment function depending on y (the dashed line) and savings function (the solid line) are displayed. We can observe the two curves intersect each other in three points which are the equilibria of commodity market. The middle point is the equilibrium in commodity market and it also is the point of steady state with zero

inflation. The first point on the left hand side is the depressive equilibrium in which deflation occurs in the economy. The third intersection is an equilibrium connected with positive inflation and we call it the point of booming equilibrium. The two outer points representing depressive and booming equilibrium are just partial equilibria because only commodity market clears.

Now, let's assume that the economy can be represented by equations (25) and (26) with stable interest rate 0.05. We calibrate such an economy and the dynamics of its variables are shown in Figures 3 and 4.

Figure 3 Evolution of production, $R=0.05$
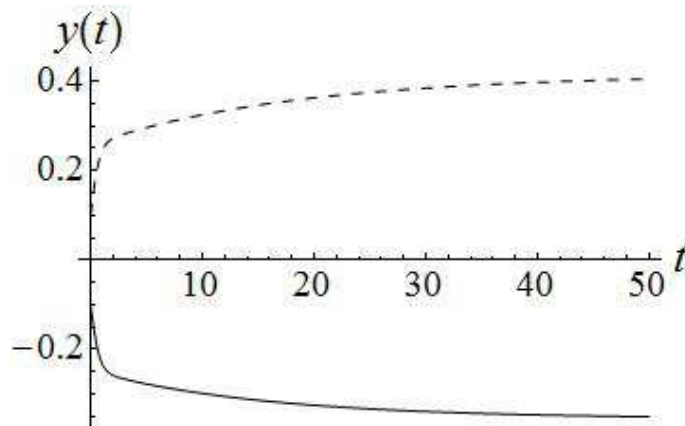
Figure 4 Evolution of inflation, $R=0.05$
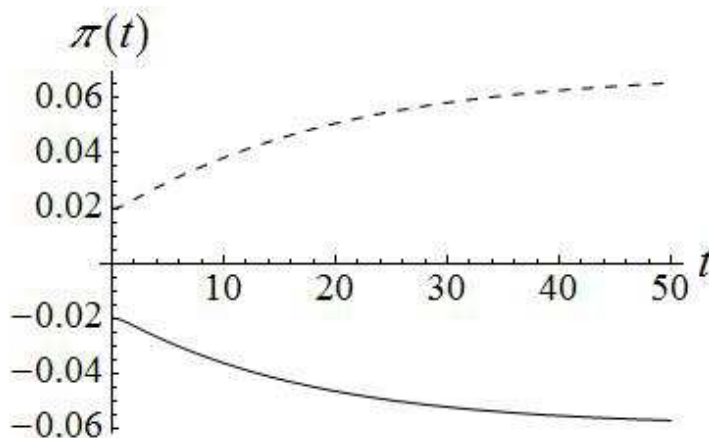
It can be seen that the trajectories of production converges to stable points of either depressive equilibrium or booming one depending on initial conditions. The dashed lines are the evolutions of production and inflation starting from some positive initial conditions of production and inflation and end up in the booming equilibrium. On the contrary, the solid lines mark out the evolutions of production and inflation (deflation) converging towards the depressive equilibrium.

Figures 5 and 6 display the trajectories of production and inflation in the case when central bank continuously manages interest rate according to Taylor rule. By substituting expression (24) into equations (25) and (26) and using the set of parameters' values described above, we obtain these two equations:

$$\dot{y}(t) = \omega\big[J - 0.12 - 0.8y(t) - 1.6\big(0.05 - \pi(t)\big)\big],$$

$$\dot{\pi}(t) = 0.1\Big\{0.1\Big[\frac{1}{1 - 0.4}y(t)\Big] - \pi(t)\Big\}$$

As the central bank continuously changes the interest rate according to modified Taylor rule shown in (24) and with calibrating parameters $\bar{R} = 0.05$, $\theta_y = 0.05$, $\theta_\pi = 4.05$, the trajectories of the variables of interest are much more complex than in previous case and either under-shooting of production and over-shooting of inflation can occur.

Figure 5 Evolution of production with R continuously managed by CB
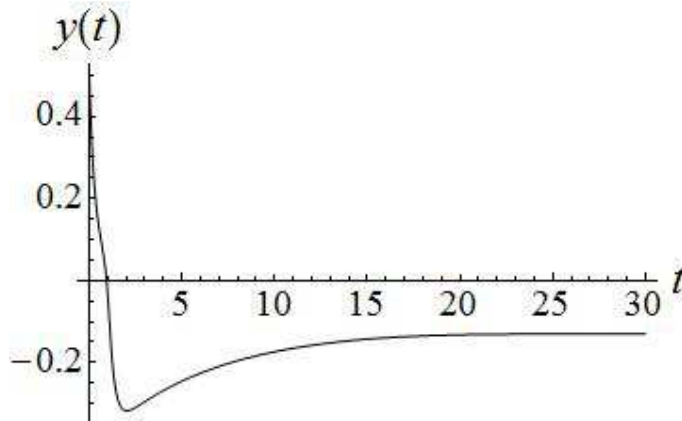
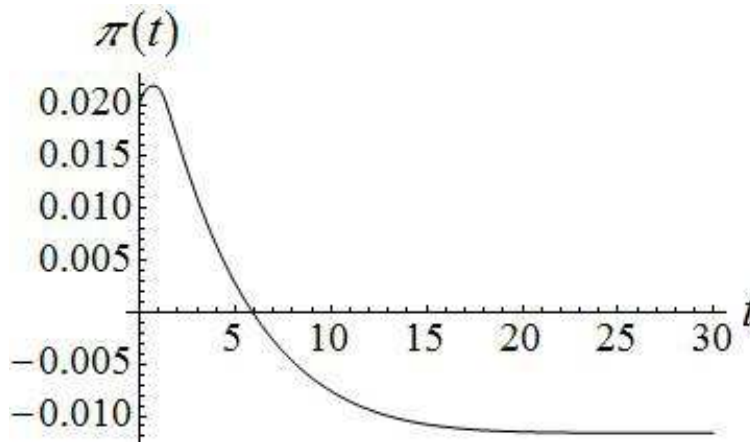**Figure 6** Evolution of inflation when R is continuously managed by CB

Figure 7 displays the evolution of interest rate managed continuously by the central bank. In figures 5, 6 and 7, two variables of our interest $y(t)$ and $\pi(t)$ as well as the interest rate start at their relatively high values, then they drop toward their equilibrium values. According to the sign of these values, one may think that it must be the stable depressive equilibrium linked with deflation (see Figure 2). We have experimented with several sets of parameters' values and it turns out that the economy always ends up in this point. The answer to the question why it is so will be the subject of further research.

Figure 7 Evolution of R continuously managed by CB
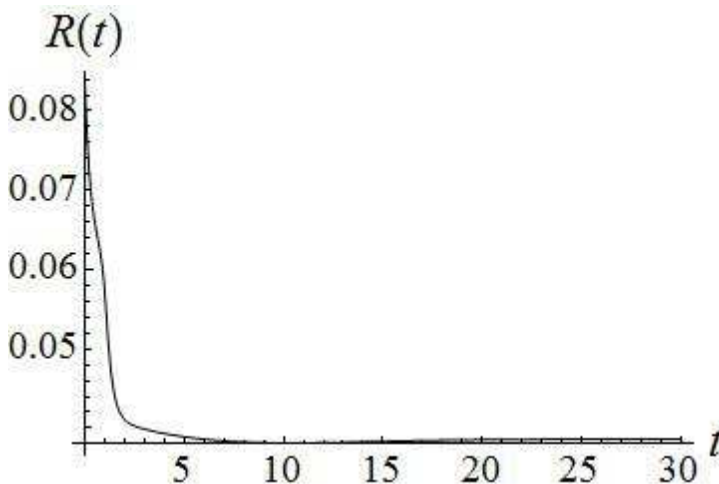
Conclusion

In this work, we examine monetary policy conducted by a central bank as an optimal control problem. In order to do so, we have proposed a continuous alternative to the traditional discrete version of the New Keynesian model. We derive two principal equations of the continuous version of the model and use loss function and a modified Taylor-type rule to study the dynamics of the whole system. To find a solution of the system and analyse its behaviour, we select a set of suitable values for all parameters included in the model. Then we use Mathematica to solve this system of equations.

The results of our experiments have confirmed the fact that monetary policy operating under two different regimes, one with the loss function, the other with the modified Taylor rule as a reaction function, leads to a different outcome. Though from the mathematical structure of the dynamics of the system, i.e. its dynamics IS curve and Phillips curve, and from its formal characteristics it should be the same, it turns out to not be the case. In the experiment with loss function, we have a continuous optimal control problem with bang-bang optimal control principle. Using the Taylor rule, the experiment has become a control problem. In order to analyse it we selected a set of suitable values for all parameters included in the model. The solution we have found differs from each other not only in its magnitude, but also more importantly the control principle are different.

The results also show that monetary policy continuously managed interest rate brings relatively more complex dynamics in comparison with the case when loss function is used and the problem was analysed as a continuous optimal control problem. Further, in both analysed problems we have also found that using a relatively simple decision rule on interest rate sometime can bring unexpected results.

References

[1] Andrle, M. – Hlédik, T. – Kameník, O. – Vlček, J. (2009): *Implementing the New Structural Model of the Czech National Bank.* CNB Working Papers No 2.

[2] Barro, R. J. – Gordon, D. B. (1983): *Rules, Discretion and Reputation in a Model of Monetary Policy.* Journal of Monetary Economics, 1983, vol. 12, no. 1, pp. 101-121.

[3] Chiarella, C. (1990): *The Elements of a Nonlinear Theory of Economic Dynamics*. Berlin, Springer Verlag, 1990.

[4] Chow, G. C. (1976): *Control Methods for Macroeconomic Policy Analysis*. The American Economic Review, 1976, vol. 66, no. 2, pp. 340-345.

[5] Driankov, D. – Hellendoorn, H. – Reinfrank, M. (1996): *An Introduction to Fuzzy Control.* Berlin, Springer Verlag, 1996.

[6] Galí, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*, Princeton, Princeton University Press, 2008.

[7] Grass, D. – Caulkins, J. P. – Feichtinger, G. – Tragler, G. – Behrens, D. A. (2008): *Optimal Control of Nonlinear Processes.* Berlin, Springer-Verlag, 2008.

[8] Jahn, J. (2007): *Introduction to the Theory of Nonlinear Optimization.* Berlin, Springer-Verlag, 2007.

[9] Kodera, J. – Málek, J. (2008): *Anomálie v nelineárním a dynamickém modelu IS-LM za přítomnosti kreditního rizika*. In: Lukáš, L. (ed.). *Výpočtová ekonomie – sborník 3. semináře.* Plzeň, Západočeská univerzita, 2008, pp. 25-43.

[10] Kukal, J. – Tran, V. Q. (2013): *A Monetary Policy Rule Based on Fuzzy Control in an Inflation Targeting Framework*. Forthcoming issue of Prague Economic Paper.

[11] Levin, A. T. - Williams, J. C. (2003): *Robust Monetary Policy with Competing Reference Models*. Journal of Monetary Economics, 2003, vol. 50, no. 5, pp. 945-75.

[12] Novák, V. (2000): *Základy fuzzy modelování.* Praha, BEN-Technická literatura, 2000.

[13] Orphanides, A. – Williams, J. C., (2008): *Learning, Expectations Formation, and the Pitfalls of Optimal Control Monetary Policy*. Journal of Monetary Economics, 2008, vol. 55, no. 1, pp. S80-S96.

[14] Pontryagin L. S. (1976): *Matěmatičeskaja teorija optimalnych proccessov*. Moskva, Nauka, 1976.

[15] Svensson, L. E. O. - Tetlow, R. (2005): *Optimum Policy Projections.* International Journal of Central Banking, December 2005, vol. 1, no. 3, pp. 177-207.

[16] Taylor, J. B. (1993): *Discretion versus Policy Rules in Practice*. Carnegie Rochester Conference Series on Public Policy, December 1993, vol. 39, pp. 95-214.

Monetary Policy as an Optimal Control Problem

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ABSTRACT

This paper analyses the monetary policy of a central bank in a simple deterministic and continuous dynamic non-linear New-Keynesian model with an active central bank conducting monetary policy within inflation targeting framework. To meet this purpose, first we derive two differential equations capturing the dynamics in the economy: the dynamic IS curve representing the commodity market and the Phillips curve capturing the connection between the real and nominal sectors of the economy in a continuous form. By introducing a quadratic loss function commonly used in New Keynesian Economics we get optimal control problem which solution will be analysed with the use of fuzzy control. Then we introduce a modified form of the Taylor rule and analyse the solution of the same differential equations capturing the dynamics of the economy using Taylor rule instead of loss function. The comparison of the solutions of both models will be demonstrated in examples in which the main characteristic of dynamics of production and inflation are displayed.

Key words: Deterministic continuous model; Dynamic IS curve; New-Keynesian Phillips curve; Loss function; Modified Taylor rule; Optimal control problem.

JEL classification: E58, C61.